Bifurcations in liquid crystals cells

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Liquid crystals: uses in optical devices

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Liquid crystals: generalities

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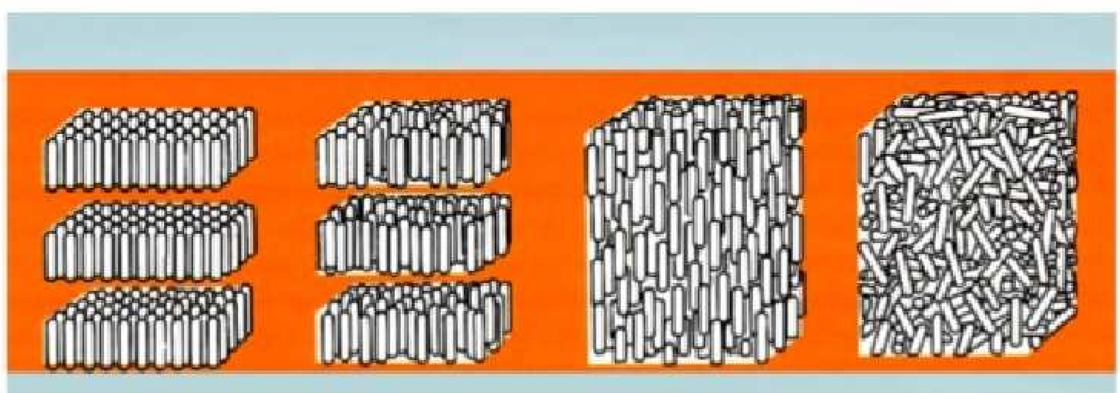
crystalline phases	mesophases	amorphous phases
3D-order	2D-, 1D-order	no long-range order



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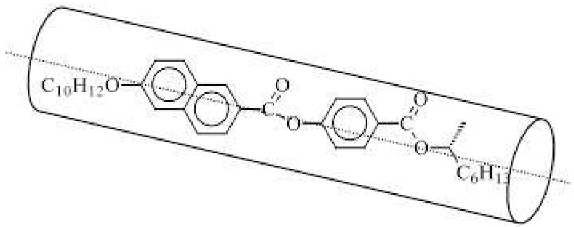

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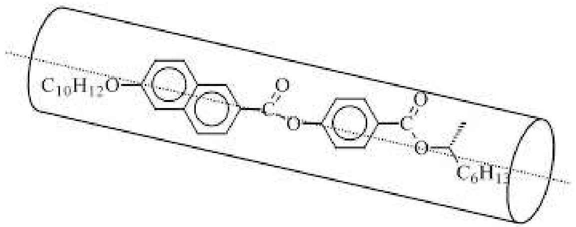

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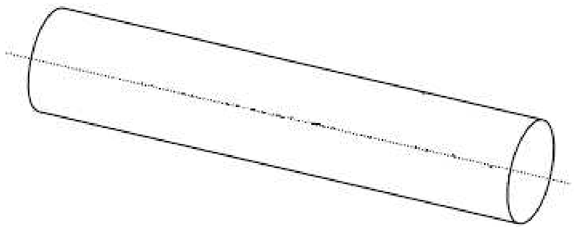

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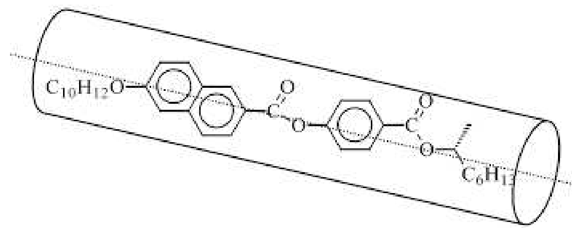

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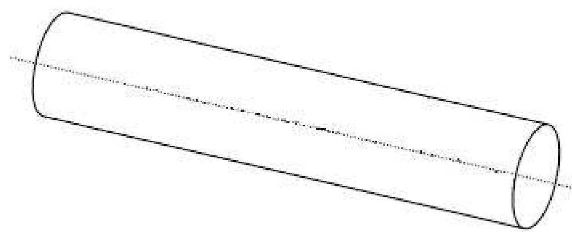
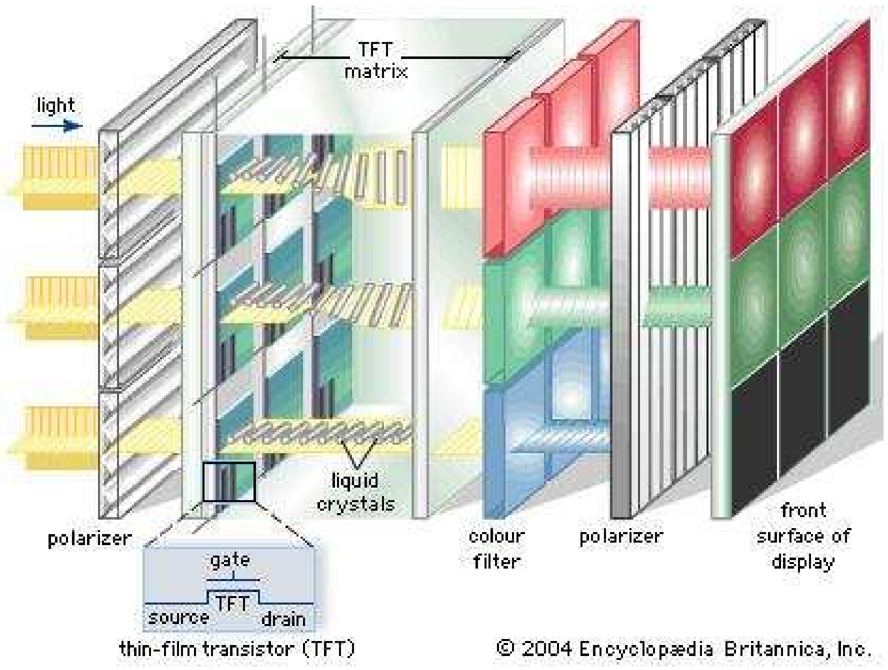


to a mathematician:

The liquid crystal cell

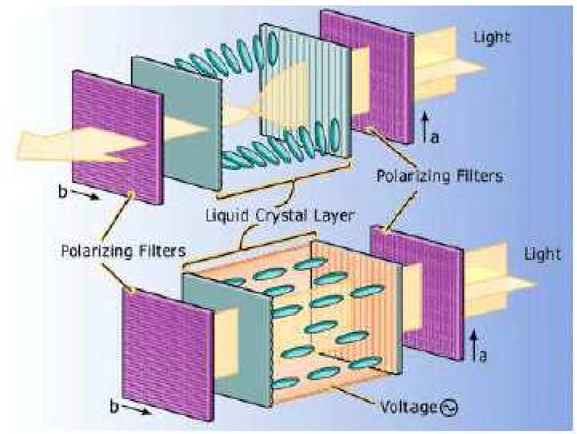
In this talk I will deal with mathematical modelling of optical phenomena in liquid crystal cells.

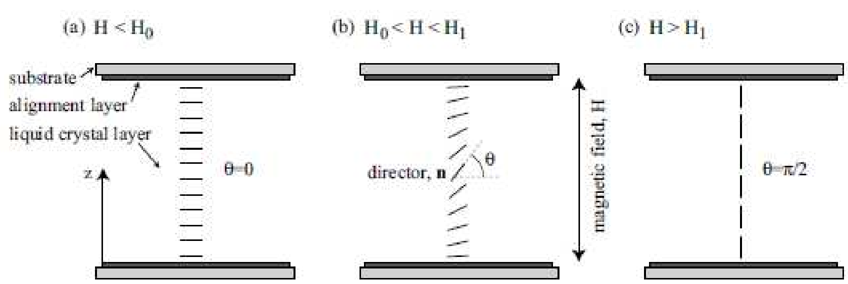
These are devices of paramount technological importance.



The Fréedericksz transition

The *Fréedericksz transition* is the simplest opto-electronic liquid crystal switch. It is the result of the competition between reorienting effects of an **applied electric or magnetic field**, the **bulk elastic energy** of the molecules and their **anchoring energy** to the bounding surfaces of a liquid crystal cell.





The mathematical setting

The **Helmholtz free energy functional** is given by

$$F = \int_{\Omega} (\sigma_F(\mathbf{n}, \nabla \mathbf{n}) + \sigma_H(\mathbf{n}))dV + \int_{\partial\Omega} \sigma_s(\mathbf{n}, \nabla \mathbf{n})dS.$$

where:

σ_F is the bulk elastic Frank-Oseen energy density

σ_H is the contribution of the applied field to the free energy

σ_s is the surface contribution to the free energy, due to anchoring of the liquid crystal molecules to the liquid crystal cells

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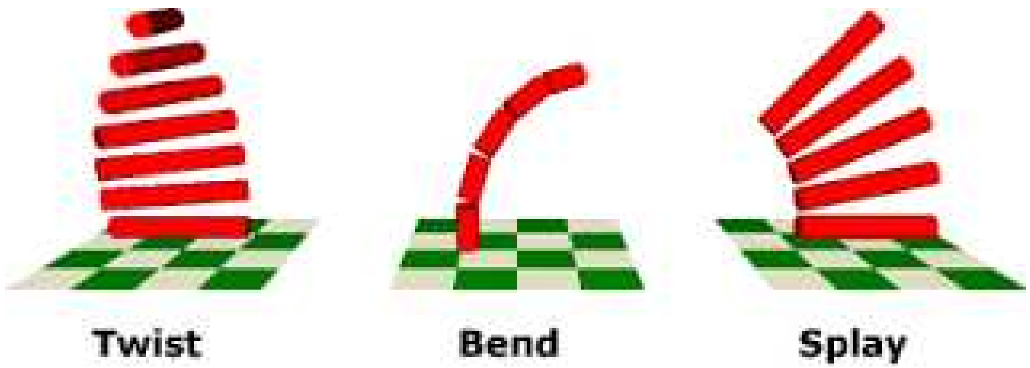
Equilibrium configurations of the LC are critical points of F.
The dynamic behaviour is governed by the gradient flow of F.
In this talk we shall consider only **equilibrium configurations**.

The three contributions to the Frank-Oseen elastic energy density σ_F of a *nematic* liquid crystal



$$k_3 \left\|\mathbf{n} \times \nabla \times \mathbf{n}\right\|^2 , \quad k_2 \left(\mathbf{n} \cdot \nabla \times \mathbf{n}\right)^2 , \quad k_1 \left(\nabla \cdot \mathbf{n}\right)^2$$

Example of anchoring conditions.

Bifurcations in twisted cells

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Strong anchoring with **twist geometry**:

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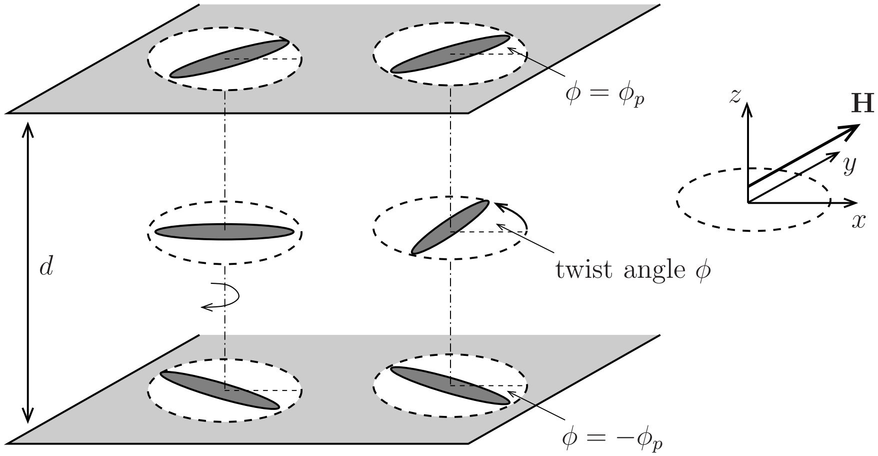
Example of anchoring conditions.

Strong anchoring with **twist geometry**:

Director $\mathbf{n} = (\cos\phi, \sin\phi, 0)$ is fixed on the cell boundary and does *not* contribute to changes in F; e.g.: in the twist-nematic cell with pre-twist at the boundary, for fixed $0 < \phi_0, \phi_1 < \frac{\pi}{2}$, the conditions are:

$$\mathbf{n}(t,0) = (\cos\phi_0, -\sin\phi_0, 0)$$

$$\mathbf{n}(t,d) = (\cos\phi_1, \sin\phi_1, 0)$$



Bifurcations in twisted cells

For the **strong anchoring** case just shown, the equilibrium equation in dimensionless form, together with the boundary condition, is the following boundary value problem with *non-homogeneous Dirichlet boundary condition:*

$$\begin{cases} 0 = \dfrac{d^2\phi}{d\zeta^2} + \lambda \sin\phi \cos\phi, & \zeta \in (0,1) \\ \phi(0) = -\phi_0, \quad \phi(1) = \phi_1 \end{cases}$$

where $\zeta := z/d$, $\lambda := \mu_0 \Delta\chi H^2 d^2 / k_2$.

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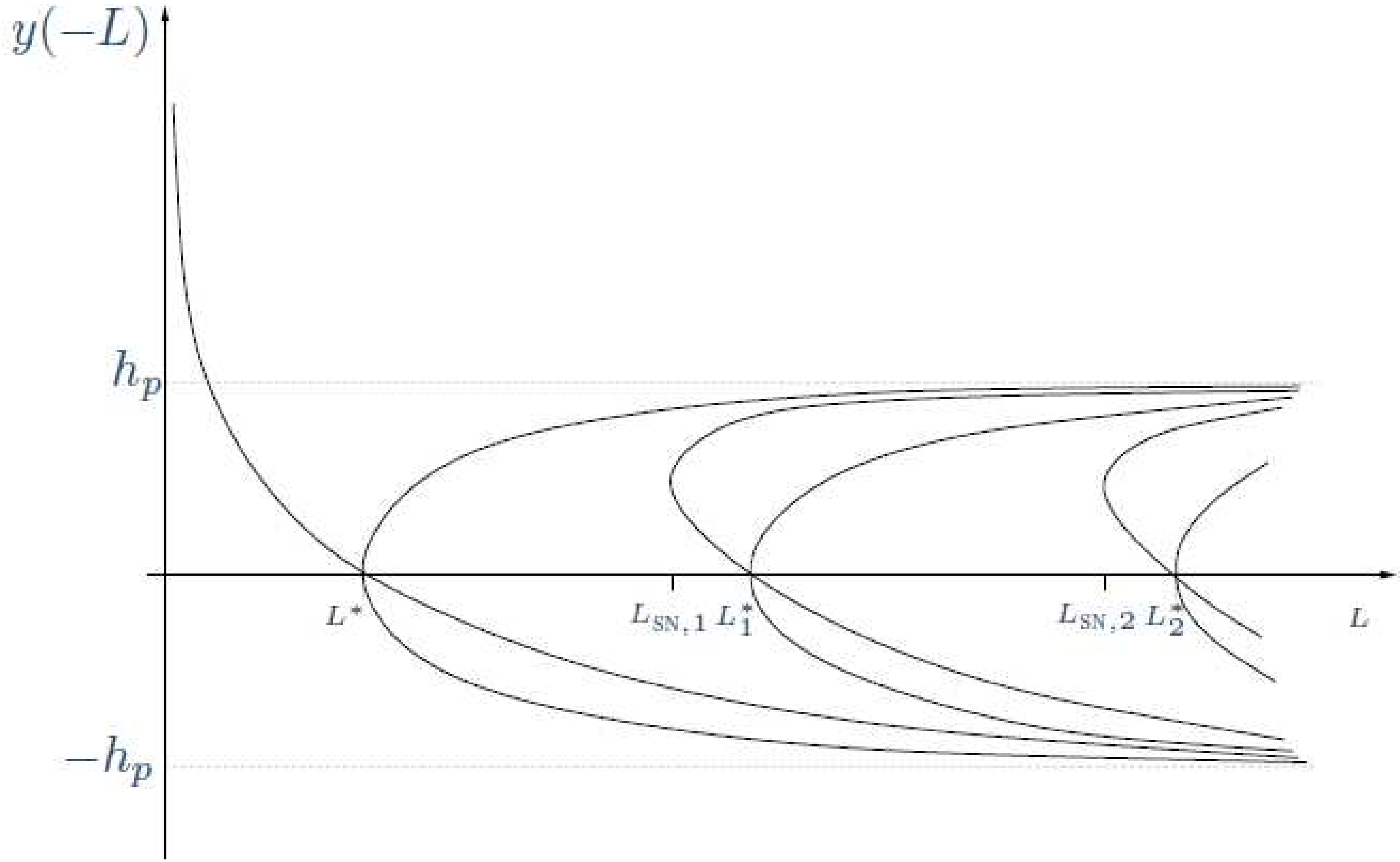
where $\zeta := z/d \;,\quad \lambda := \mu_0 \Delta\chi H^2 d^2/k_2$.

A further change of variables $\zeta \mapsto t := (\zeta - \frac{1}{2})\sqrt{\lambda/2}$, and $L = \sqrt{\lambda/8}$, transforms it into

$$\begin{cases} x' = y, \quad y' = -\sin 2x, & t \in (-L, L) \\ x(-L) = -\phi_0, \quad x(L) = \phi_1 \end{cases} \tag{1}$$

where $x(t) = \phi(\zeta(t))$.

Bifurcation diagram of (1) when $\phi_0 = \phi_1 =: \phi_p \in (0, \frac{\pi}{2})$:



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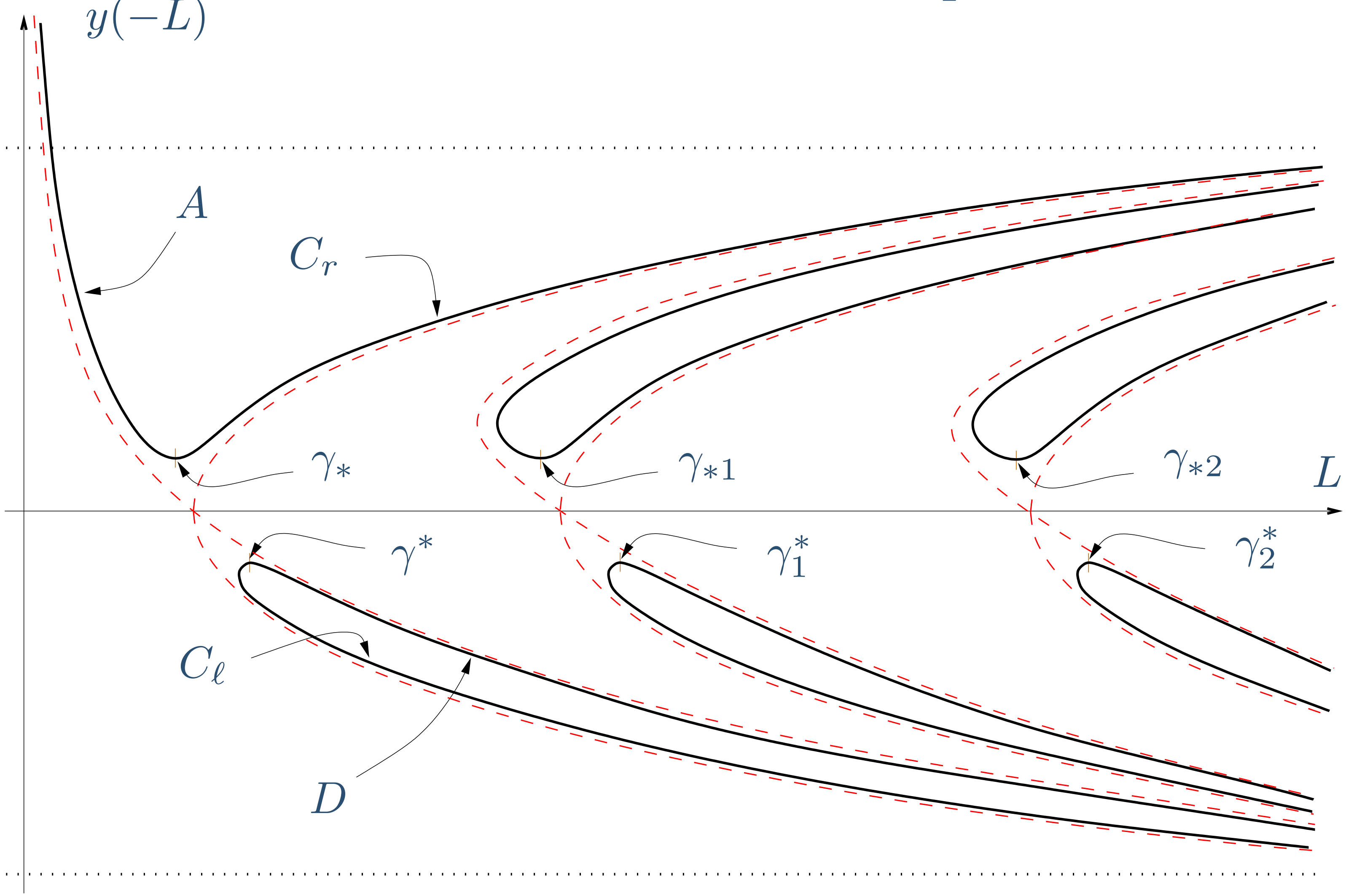
Bifurcation diagram for (1) when $0 < \phi_0 < \phi_1 < \frac{\pi}{2}$:



Tools and approach:
Time maps and phase plane analysis based on “special” solutions.

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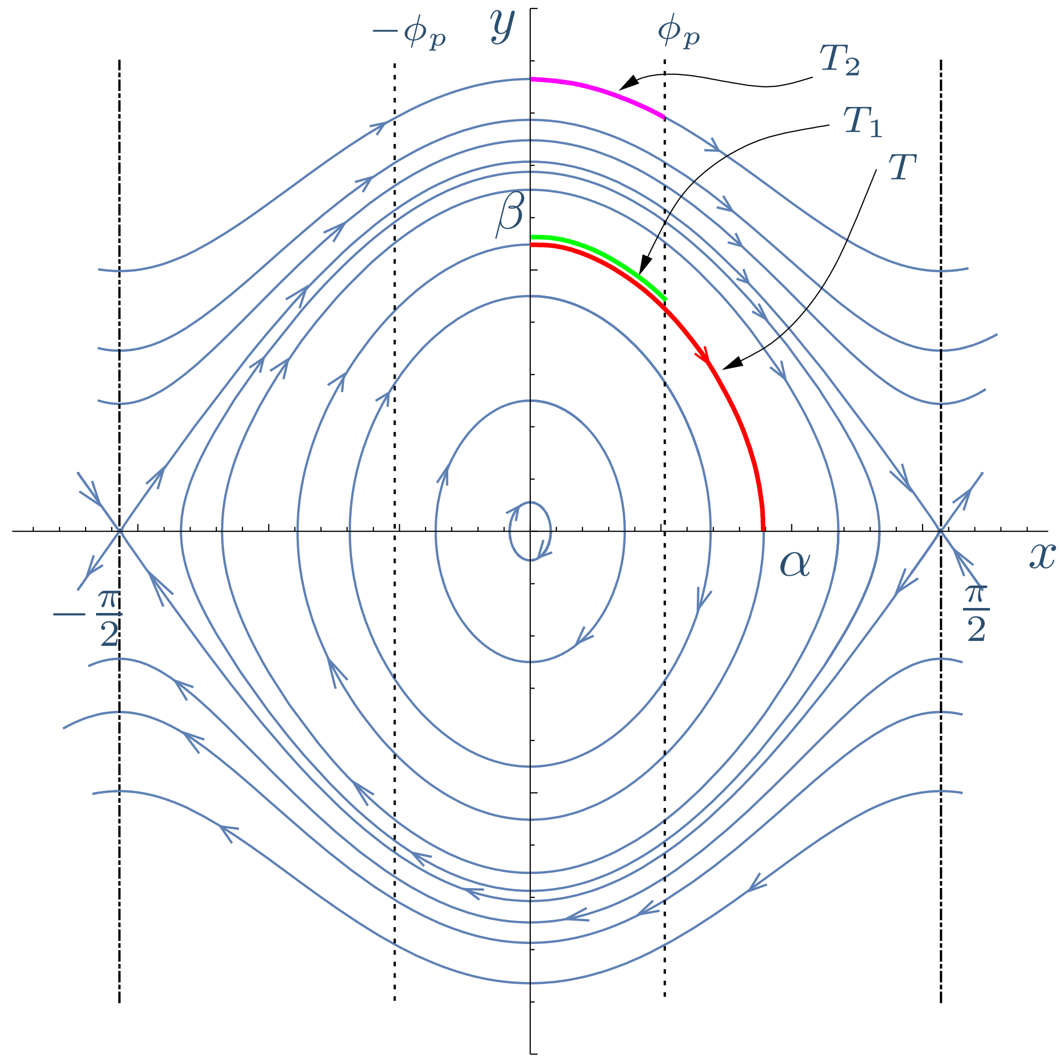

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Bifurcations in twisted cells

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$$\begin{cases} x' = y \\ y' = -\sin 2x \end{cases} \tag{2}$$

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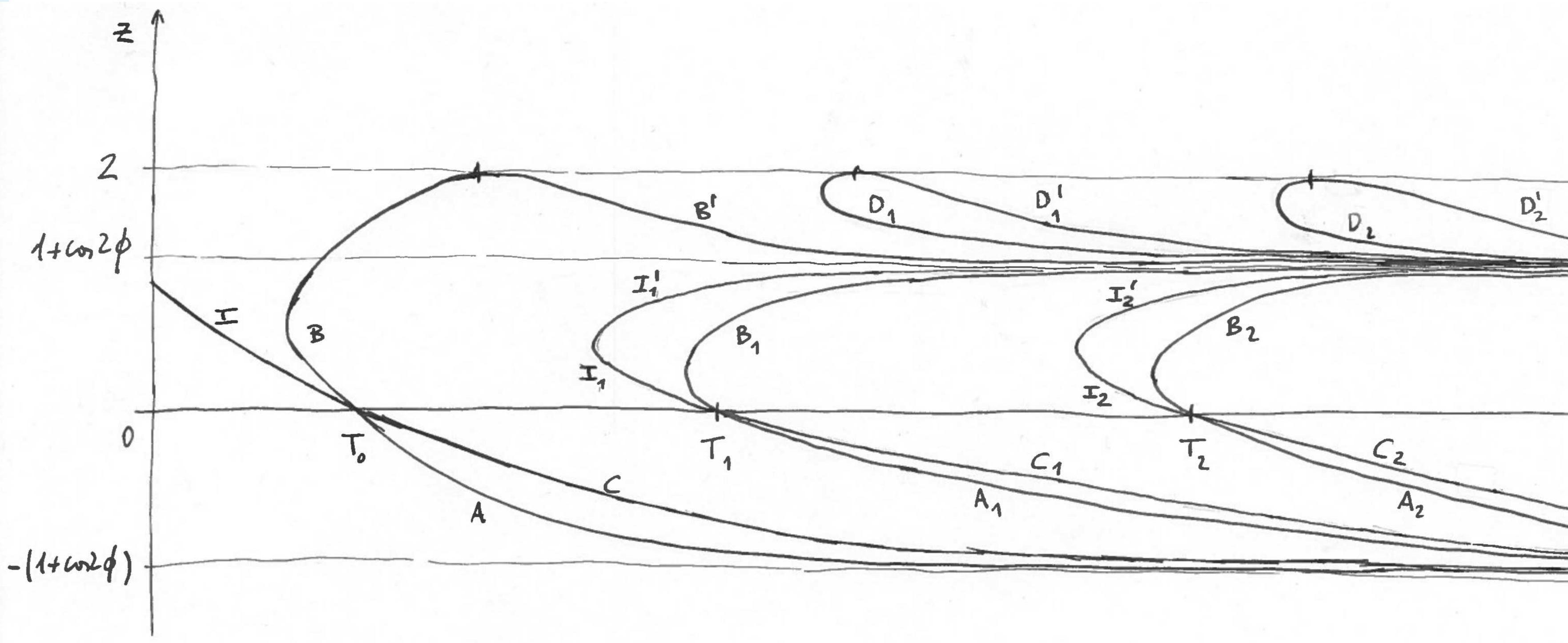
Its proof depends on some convexity property a certain time map, that still resists current attacks.

Bifurcations in twisted cells



From joint works with:

E.C. Gartland, Jr. (Kent State U, Kent OH)

M. Grinfeld, M. Langer, N.J. Mottram (U Strathclyde, Glasgow)

M.I. Méndez (IES Antonio López García, Madrid)

J.T. Pinto (IST, U Lisbon, Lisbon)

K. Xayxanadasy (NUOL, Vientiane)

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Math. Models Meth. Appl. Sci. (2007) **17**, 2009–2034.

J. Differ. Equ. (2009) **246**, 2590–2600.

Quart. Appl. Math. (2012) **70**, 99–110.

Eur. J. Appl. Math. (2009) **20**, 269–287; (2017) **28**, 243–260.

and work in progress.

Thank you!