Fluxo estocástico de um fluído viscoso e comportamento assintótico

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Ciência 2018. Lisboa, 2 de Julho de 2018

1. *Physical motivation*

2. *Existence and uniqueness of stochastic Lagrangian flow*
   - Well-posedness of the Lagrangian dynamics

3. *Inviscid limit/ Principle of large deviations*
   - Rate function
   - Variational representation
   - Tightness
   - Schilder's Theorem

Stochastic Lagrangian Navier-Stokes flow

We consider the velocity field u^ϵ given as solution of the Navier-Stokes equations in the 2-dimensional torus $\mathbb{T}^2$

$$\begin{cases} \frac{\partial u^\epsilon}{\partial t} + (u^\epsilon.\nabla)u^\epsilon = \epsilon\Delta u^\epsilon + \nabla p^\epsilon \\ \nabla.u^\epsilon = 0 \\ u^\epsilon(x,0) = u_0(x), \;\; u_0 \in H^1(\mathbb{T}^2) \end{cases}$$

$$\epsilon = \frac{1}{\mathrm{Re}} = \frac{\nu}{UL}$$

and define the stochastic Navier-stokes flows as solution of a stochastic differential equation

$$dX_t^\epsilon(x) = u^\epsilon\left(X_t^\epsilon(x), t\right) dt + \sqrt{\epsilon}\sigma(X_t^\epsilon(x))dW_t, \;\; X_0^\epsilon = x, \;\; x \in \mathbb{T}^2.$$

Then we will establish a Schilder's type theorem, for the asymptotic behaviour of the Navier-Stokes flows X_t^ϵ, when the viscosity $\epsilon \to 0$.

Deterministic Lagrangian Euler flow

We establish a large deviations principle on the Banach space $C([0,T],L^2(\mathbb{T}^2))$. As $\epsilon \to 0$, the Navier-Stokes flows will converge with an exponential rate function to the deterministic Euler flow X_t, defined as the solution of the differential equation

$$dX_t(x) = u\big(X_t(x),t\big)\ dt,\ \ X_0 = x,\ \ x \in \mathbb{T}^2,$$

with velocity field $u(x,t)$ being solution of the Euler equations

$$\begin{cases} \frac{\partial u}{\partial t} + (u.\nabla)u = \nabla p \\ \nabla.u = 0 \\ u(x,0) = u_0(x),\ \ u_0 \in H^1(\mathbb{T}^2). \end{cases}$$

Here, the drift depends on the (small) parameter ϵ that is the viscosity of the fluid, then the large deviations principle has a physical meaning corresponding to inviscid limit transition.

Brownian motion

Let $W_t^{k_1}$, $W_t^{k_2}$, $k_1, k_2 \in \mathbb{Z}$ be $\mathbb{R}-$valued independent Brownian motions, defined on a probability space $(\Omega, \mathcal{F}, P)$. We consider the stochastic process W_t defined by

$$W_t = \sum_{k=(k_1,k_2)\in\mathbb{Z}^2/\{(0,0)\}} W_t^{k_1} e_{k_1} + W_t^{k_2} e_{k_2} = (\ldots, \underbrace{W_t^{k_1}, W_t^{k_2}}_{\text{position k}}, \ldots)$$

where

$$\begin{cases} e_{k_1} = (\ldots, 0, 0, \underbrace{1, 0}_{\text{k position}}, 0, 0, \ldots) \\ e_{k_2} = (\ldots, 0, 0, \underbrace{0, 1}_{\text{k position}}, 0, 0, \ldots) \end{cases}$$

Brownian motion

The set $\left\{e_{k_1}, e_{k_2}, k \in \mathbb{Z}^2 \setminus \{(0,0)\}\right\}$ defines an orthonormal basis to the Hilbert space l^2 of all sequences $z = (z_k)$, $k \in \mathbb{Z}^2 \setminus \{(0,0)\}$, with norm

$$\|z\|_{l^2}^2 = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} |z_k|^2 < \infty$$

and inner product

$$\langle z, y\rangle_{l^2} = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} z_k y_k, \quad z = (z_k)\,, \;\; y = (y_k)\,.$$

Thus, the stochastic process W_t can be understood as a cylindrical Wiener process over l^2. W_t can be considered as a Q-Wiener process in a bigger Hilbert space l^2_* in which l^2 is embedded through a Hilbert-Schmidt operator. *We will consider on the probability space* $(\Omega, \mathcal{F}, P)$ *the filtration* $\{\mathcal{F}_t\}$ *induced by the process* W_t.

diffusion operator

We are interested in the particular case of equation, where the diffusion operator σ is formally represented by the infinite dimensional matrix

$$\sigma(x) = \begin{pmatrix} \dots, A_k^1(x), B_k^1(x), \dots \\ \dots, A_k^2(x), B_k^2(x), \dots \end{pmatrix}$$

where

$$A_k^1(x) = \frac{k_2 \cos(k.x)}{|k|^\beta}, \quad A_k^2(x) = -\frac{k_1 \cos(k.x)}{|k|^\beta}$$
$$B_k^1(x) = \frac{k_2 \sin(k.x)}{|k|^\beta}, \quad B_k^2(x) = -\frac{k_1 \sin(k.x)}{|k|^\beta}$$

with $k = (k_1, k_2) \in \mathbb{Z}^2/\{(0,0)\}$ and $\beta > 3$.

$$\sigma : \mathbb{T}^2 \to HS(l^2, \mathbb{R}^2)$$
$\sigma(x)e_{k_i}$ are divergence free vector fields

Evolution equation

Let us introduce the following stochastic differential equation,

$$dX_t(x) = u(X_t(x), t)dt + \sigma(X_t(x))dW_t, \;\; X_0 = x, \;\; x \in \mathbb{T}^2,$$

on the 2-dimensional torus $\mathbb{T}^2 \equiv [0, 2\pi] \times [0, 2\pi]$ under periodic boundary conditions.
The drift $u \in L^2\left(0, T;\, W^{1,p}(\mathbb{T}^2)\right)$, $p > 1$,
the diffusion operator

$$\sigma : \mathbb{T}^2 \to HS(l^2, \mathbb{R}^2)$$

satisfying appropriate integrability conditions.

Evolution equation

(V. I. Arnold, Ann. Inst. Fourier 1966)

$$S[X] = \frac{1}{2} \int_0^T \|\dot{X}(t)\|_{L^2}^2 dt$$

(A. B. Cruzeiro, C., cmp 2007)

$$S[X] = \frac{1}{2} E \int_0^T \| \lim_{\Delta t \to 0} \frac{1}{\Delta t} E^{\mathcal{F}_t} (X_{t+\Delta t} - X_t) \|_{L^2}^2 dt$$

Definition of solution (in the DiPerna Lions sense)

Let $X_t(\omega, x)$ be a measurable stochastic field defined in $\mathbb{R}_+ \times \Omega \times \mathbb{T}^2$. X_t is a a.e. stochastic incompressible flow solution of the equation, if

(a) for a.e.$-x \in \mathbb{T}^2$, $X_t(x)$ is a continuous stochastic process adapted to the filtration $\{\mathcal{F}_t\}$ such that for any $T > 0$

$$\int_0^T |u(X_s(x), s)|ds + \int_0^T \|\sigma(X_s(x))\|_{HS}^2 ds < \infty, \quad \text{a.e.} - \omega$$

and $X_t(x)$ solves

$$X_t(x) = x + \int_0^t u(X_s(x), s)\, ds + \int_0^t \sigma(X_s(x))\, dW_s, \ \ \forall t \in [0, T];$$

(b) for every $t \geq 0$, and a.e.$-\omega$, the map $X_t(\omega, \cdot) : \mathbb{T}^2 \to \mathbb{T}^2$ preserves the Lebesgue measure

$$\int_{\mathbb{T}^2} \varphi(X_t(x))\, dx = \int_{\mathbb{T}^2} \varphi(x)\, dx$$

Well-posedness of the Lagrangian dynamics

Theorem

Let $u \in L^2(0, T;\ H^1(\mathbb{T}^2))$ *such that* $div\ u = 0$. *Then, there exists a unique stochastic incompressible a.e. flow* X_t *defined almost surely in the variables* ω *and* x *that solves*

$$dX_t = u(X_t, t)dt + \sigma(X_t)dW_t,\ \ X_0 = x.$$

Moreover for a.e$-\omega$ *the path* $X(\omega, \cdot)$ *belongs to* $C([0, T], L^2(\mathbb{T}^2))$.

(G. Crippa, C. De Lellis, 2008; X. Zhang, 2010)

Crucial estimate to prove the theorem

Let X_t^n and X_t^m be the solutions to the regularized stochastic differential equation with coefficients (u_n, σ_n) and (u_m, σ_m), respectively. Then for all $T > 0$ and $\delta > 0$, we have

$$\mathbb{E}\int_{\mathbb{T}^2} \log\left(\frac{\sup_{t\in[0,T]}|X_t^n(x)-X_t^m(x)|^2}{\delta^2}+1\right)dx$$
$$\leq \frac{C_1}{\delta}\Bigg(\int_0^T\int_{\mathbb{T}^2}|u_n(x,t)-u_m(x,t)|dxdt + \left\{\int_0^T\int_{\mathbb{T}^2}\|\sigma_n(x)-\sigma_m(x)\|_{HS}^2 dxdt\right\}^{\frac{1}{2}} + \frac{1}{\delta}\int_0^T\int_{\mathbb{T}^2}\|\sigma_n(x)-\sigma_m(x)\|_{HS}^2 dxdt\Bigg) + C_2, \quad \text{with}$$

$$C_2 = C\left(1+\int_0^T\int_{\mathbb{T}^2}|\nabla u_m(x,t)|^2dxdt + \left\{\int_0^T\int_{\mathbb{T}^2}|||\nabla\sigma_n(x)|||^2dxdt\right\}^{\frac{1}{2}} + \int_{\mathbb{T}^2}|||\nabla\sigma_m(x)|||^2dx\right),$$

Large deviation principle/Laplace-Varadhan principle

Given a Banach space Y, a $Y-$valued random variables $\{X^\epsilon\}$ satisfies a large deviation principle on Y with rate function I (the level sets $\{f \in Y : I(f) \leq M\}$, $M > 0$, are compact sets of Y) if for every Borel set Γ of Y, we have

$$\begin{aligned} -\inf_{f\in \mathrm{int}(\Gamma)} I(f) &\leq \liminf_{\epsilon\to 0} \epsilon \log \mathbb{P}\{X^\epsilon \in \mathrm{int}(\Gamma)\} \\ &\leq \limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}\{X^\epsilon \in \bar{\Gamma}\} \leq -\inf_{f\in\bar{\Gamma}} I(f). \end{aligned}$$

$$\mathbb{P}\{X^\epsilon \in \Gamma\} \approx e^{-\frac{1}{\epsilon}\inf_{f\in\Gamma} I(f)}$$

The large deviation principle is equivalent to the following Laplace-Varadhan principle:

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left\{e^{-\frac{1}{\epsilon}g(X^\epsilon)}\right\} = -\inf_{f\in Y}\{g(f) + I(f)\}$$

for all bounded continuous function $g : Y \to \mathbb{R}$.

Rate function

We define the mapping $I : C\big([0,T], L^2(\mathbb{T}^2)\big) \mapsto \mathbb{R}$ as

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathbb{H}:\, S(h) = f\}} ||h||^2_{\mathbb{H}},$$

where

$$S(h) := X^h_t(x)$$

is the unique solution the the following deterministic control equation

$$dX^h_t(x) = x + \int_0^t \langle \sigma\big(X^h_s(x)\big), \dot{h}(s)\rangle_{l^2} ds + \int_0^t u\big(X^h_s(x), s\big) ds.$$

with

$$h \in \mathbb{H} = \{h : [0,T] \to l^2 : \|h\|^2_{\mathbb{H}} = \int_0^T \|\dot{h}(s)\|^2_{l^2} ds < \infty\}.$$

The function I is a rate function!

Variational representation

Let $X_t^\epsilon(x)$ be the unique solution of the following stochastic differential equation

$$dX_t^\epsilon(x) = u^\epsilon\left(X_t^\epsilon(x), t\right) dt + \sqrt{\epsilon}\sigma(X_t^\epsilon(x))dW_t, \;\; X_0^\epsilon = x, \;\; x \in \mathbb{T}^2.$$

For any bounded function $f : C\left([0, T], L^2(\mathbb{T}^2)\right) \to \mathbb{R}$, the following variational representation hold (Budhiraja A., Dupuis P., 2000)

$$-\epsilon \log \mathbb{E}(e^{-\frac{1}{\epsilon} f(X_t^\epsilon)}) = \inf_{h \in A} \mathbb{E}\left(f(X_t^{\epsilon,h}) + \frac{1}{2}||h||^2_{\mathbb{H}} \right)$$

where A is the set of l^2-valued, $\mathcal{F}_t$-predictable stochastic processes such that

$$\mathbb{E}\left(\int_0^T ||\dot{h}(s)||^2_{l^2} ds \right) < \infty,$$

and $X_t^{\epsilon,h}$ is the unique solution of

$$dX_t^{\epsilon,h} = u^\epsilon\left(X_t^{\epsilon,h}, t\right)dt + \langle \sigma\left(X_t^{\epsilon,h}\right), \dot{h}_t \rangle_{l^2} dt + \sqrt{\epsilon}\sigma\left(X_t^{\epsilon,h}\right) dW_t, \;\; X_0^h = x \in \mathbb{T}^2.$$

Tightness

Assume that $\{h^\epsilon, \ \epsilon > 0\} \subset A_N$ and consider the corresponding solutions $X_t^{\epsilon,h^\epsilon}$, $\epsilon > 0$, of stochastic control equation. Then:

1. The laws of $\{(h^\epsilon, X_t^{\epsilon,h^\epsilon}, W\}$ are tight in $B_N \times C\big([0,T], L^2(\mathbb{T}^2)\big) \times C([0,T], l^2_*)$.
2. There exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, a sequence of stochastic processes $\{(\tilde{h}^\epsilon, \tilde{X}_t^{\epsilon,\tilde{h}^\epsilon}, \tilde{W}^\epsilon)\}$ and a stochastic process $\{(h, X_t^h, \tilde{W})\}$ defined in this probability space taking values in $B_N \times C\big([0,T], L^2(\mathbb{T}^2)\big) \times C([0,T], l^2_*)$ such that:

(a) $\{(\tilde{h}^\epsilon, \tilde{X}_t^{\epsilon,\tilde{h}^\epsilon}, \tilde{W}^\epsilon)\}$ has the same law as $\{(h^\epsilon, X_t^{\epsilon,h^\epsilon}, W)\}$ for every ϵ;

(b) $\{(\tilde{h}^\epsilon, \tilde{X}_t^{\epsilon,\tilde{h}^\epsilon}, \tilde{W}^\epsilon)\} \to \{(h, X_t^h, \tilde{W})\}$ in $B_N \times C\big([0,T], L^2(\mathbb{T}^2)\big) \times C([0,T], l^2_*)$ a.e.$-\tilde{P}$, when $\epsilon \to 0$;

(c) for a.e.$-\tilde{P}$, X^h is the solution of the deterministic control equation.

Schilder's Theorem

Using the weak convergence approach developed by (P. Dupuis and R. S. Ellis)

Theorem

Let $\{X^\epsilon\}$ be the solutions of

$$dX_t^\epsilon(x) = u^\epsilon\left(X_t^\epsilon(x), t\right) dt + \sqrt{\epsilon}\sigma(X_t^\epsilon(x))dW_t, \;\; X_0^\epsilon = x, \;\; x \in \mathbb{T}^2.$$

Then $\{X^\epsilon\}$ satisfies the Laplace's Principle in $C\left([0,T], L^2(\mathbb{T}^2)\right)$ with the rate function $I(f)$.

Tightness criterion on $C([0,T],L^p(S))$

Assume that S is a bounded open subset of $\mathbb{R}^d$. A family of probability measures $\{P^\epsilon, \epsilon > 0\}$ on the space $C([0,T],L^p(S))$, $p \geq 1$, is tight if for all $\rho > 0$ the following properties hold

1) $\lim_{r\searrow 0} \sup_{\epsilon>0} P^\epsilon \Big\{ f \in C([0,T],L^p(S)) :$

$$\sup_{0\leq t\leq T} \frac{1}{|B_r(0)|} \int_{B_r(0)} \int_S |f(t,x) - f(t,x+z)|^p \, dxdz \geq \rho \Big\} = 0,$$

2) $\lim_{M\nearrow\infty} \sup_{\epsilon>0} P^\epsilon \Big\{ f \in C([0,T],L^p(S)) :$

$$\sup_{0\leq t\leq T} \|f(t,\cdot)\|^p_{L^p(S)} \geq M \Big\} = 0,$$

3) $\lim_{\delta\searrow 0} \sup_{\epsilon>0} P^\epsilon \Big\{ f \in C([0,T],L^p(S)) :$

$$\sup_{\substack{|s-t|\leq\delta \\ 0\leq t\leq T}} \|f(s,\cdot) - f(t,\cdot)\|^p_{L^p(S)} \geq \rho \Big\} = 0.$$

Tightness criterion on $C([0,T],L^2(\mathbb{T}^2))$

Lemma

Let $X_t^{\epsilon,h^\epsilon}$, $\epsilon>0$, be the solutions to the stochastic control equation. Then there exists a continuous function g, independent on ϵ, such that

$$\lim_{r\to 0} g(r)=0$$

and

$$\mathbb{E}\left\{\frac{1}{|B_r(0)|}\int_{B_r(0)}\int_{\mathbb{T}^2}\sup_{t\in[0,T]}|X_t^{\epsilon,h^\epsilon}(x)-X_t^{\epsilon,h^\epsilon}(x+z)|^2\,dxdz\right\}\le g(r)$$

for every $z\in\mathbb{T}^2$.

Tightness criterion on $C([0,T], L^2(\mathbb{T}^2))$

Lemma

Let $X_t^{\epsilon,h^\epsilon}$, $\epsilon > 0$, *be the solutions to the stochastic control equation. Then there exists a continuous function* f *independent of* ϵ *such that*

$$\lim_{\tau\to 0} f(\tau) = 0$$

and for any $r, t > 0$ *satisfying* $0 \le t \le t + r \le T$, *we have*

$$\mathbb{E}\int_{\mathbb{T}^2} \sup_{t\in[0,T-r]} |X_{t+r}^{\epsilon,h^\epsilon}(x) - X_t^{\epsilon,h^\epsilon}(x)|^2\,dx \le f(|r|).$$

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