Mathematical Modelling of Neural Activity

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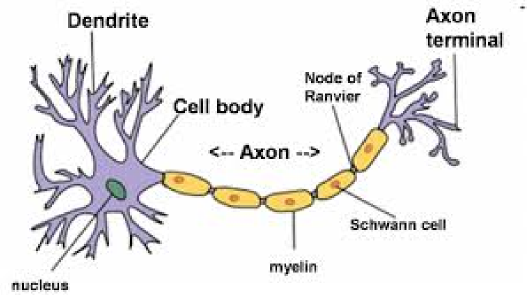
OUTLINE OF THE TALK

1. **Introduction**
2. **Numerical Methods for the Deterministic Equation**
3. **Numerical results**
4. **Ongoing work**
5. **Stochastic Neural Field Equation**
6. **Research Project NEUROFIELD**

INTRODUCTION -THE HUMAN BRAIN

According to a lower estimate from 2009, the human nervous system contains 0.89×10^{11} neurons, which are connected by about 10^{15} synapses.



COMMUNICATION BETWEEN NEURONS

The change of voltage in the cell membrane of a neuron results in a voltage spike called an action potential, which triggers the release of other neurotransmitters. That is, neurons communicate with each other by firing.

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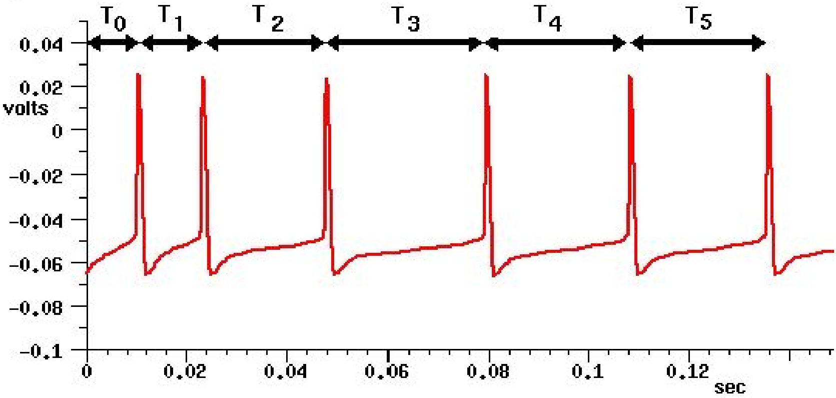
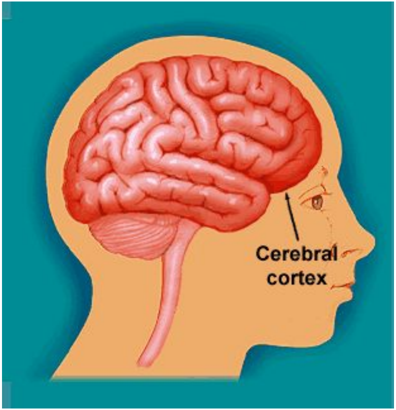


THE CEREBRAL CORTEX

The cerebral cortex is the brain's outer layer of neural tissue in humans and other mammals.

It plays a key role in controlling memory, attention, perception, awareness, thought, language and other important processes.

The cortex of a human is about 2-4 mm thick and contains about one fifth of all the neurons.



MATHEMATICAL MODELS

- Hodgkin and Huxley, 1952 - the most successful mathematical model describing the mechanism of ion currents and voltage changes in the neuron membrane. It consists of a system of 4 ordinary differential equations.
- FitzHugh-Nagumo Equations, 1962 - the Hodgkin-Huxley system was reduced to a system of two equations; it describes the propagation of impulses in the nervous system.
- Neural Field Equations - introduced by Wilson and Cowan, in 1972, and Amari, in 1977. The main idea of the Neural Field Models is to treat the cortex as a continuous space and describe the spatiotemporal dynamics of the neural interactions.

APPLICATIONS OF NEURAL FIELDS

- In Neuroscience - interpretation of experimental data, including information obtained from EEG, fMRI and optical imaging.
- In Robotics - the architecture of autonomous robots, able to interact with other agents in solving a mutual task, is strongly inspired by the processing principles and the neuronal circuitry in the primate brain.

NEURAL FIELD EQUATION

$$c\frac{\partial}{\partial t}V(\bar{x},t) = I(\bar{x},t) - V(\bar{x},t) + \int_{\Omega} K(\|\bar{x}-\bar{y}\|_2)S(V(\bar{y},t))d\bar{y}, \quad (1)$$

$$t \in [0, T], \bar{x} \in \Omega \subset \mathbb{R}^2;$$

Initial Condition: $V(\bar{x}, 0) = V_0(\bar{x}), \qquad \bar{x} \in \Omega.$

- $V(\bar{x}, t)$ - the membrane potential in point $\bar{x}$ at time t;
- $I(\bar{x}, t)$ - external sources of excitation;
- $S(V)$ - dependence between the firing rate of the neurons and their membrane potentials (sigmoidal or Heaviside function);
- $K(\|\bar{x}-\bar{y}\|_2)$ - connectivity between neurons at $\bar{x}$ and $\bar{y}$.

NUMERICAL EXAMPLE 1

Connectivity Kernel:

$$K(r) = \frac{1}{\sqrt{2\pi\xi_1^2}} \exp\left(-\frac{r^2}{2\pi\xi_1^2}\right) - \frac{A}{\sqrt{2\pi\xi_2^2}} \exp\left(-\frac{r^2}{2\pi\xi_2^2}\right),$$

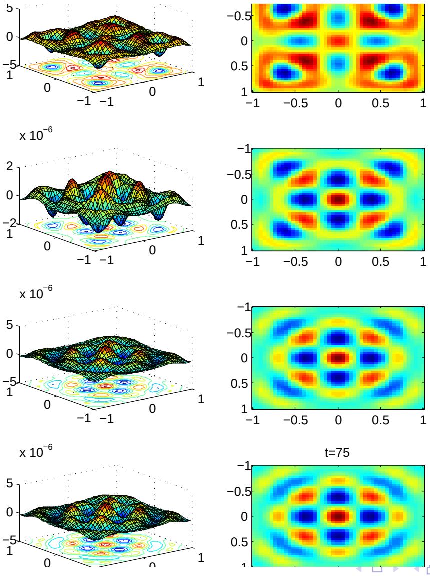
where A, ξ_1, ξ_2 - given positive numbers.
External input: $I \equiv 0$. Firing rate function: $S(x) = \frac{2}{1+e^{-\mu x}}, \quad \mu > 0.$
Propagation speed: no delay, $v = 1$.
Initial condition:

$$V_0(x_1, x_2) \equiv 0.01,$$

$\forall \bar{x} \in \Omega, t \in [-\tau_{max}, 0]$.

NUMERICAL EXAMPLE 1

$\xi_1 = 0.1$, $xi_2 = 0.2$, $A = 1$; $\mu = 45$; no delay; time=20, 30, 40, 50.



NUMERICAL EXAMPLE 2 (WORKING MEMORY)

Connectivity Kernel:

$$K(r) = A\exp(-kr)\left((k\sin(\alpha_1 r) + \cos(\alpha_1 r)\right),$$

where $r = \sqrt{x^2+y^2}$, $A = 0.02$, $k = 0.8$, $\alpha_1 = 1$.

External input:

- If $0 < t < 1.5$, $I(x,t) = 0.1\exp(-(x-5t)^2) + 0.1\exp(-(x-10)^2) + 0.1\exp(-(x+10)^2)$;
- If $1.5 \leq t < 3$, $I(x,t) = 0.1\exp(-(x-5t)^2)$;
- If $3 \leq t < 4.5$, $I(x,t) = 0.1\exp(-(x-5t)^2) + 0.1\exp(-(x-10)^2) + 0.1\exp(-(x+10)^2)$;
- If $t > 4.5$, $I(x,t) = 0$.

Firing rate function: $S(x) = H(x-0.1)$

Initial condition:

$$V_0(x_1, x_2) = 0$$

NUMERICAL EXAMPLE2 (WORKING MEMORY)

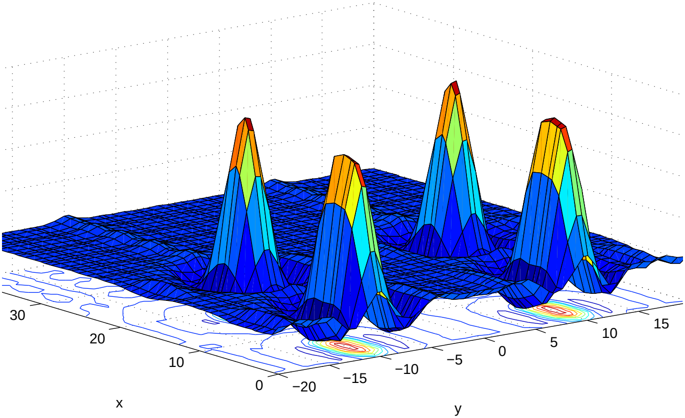


Figure: Solution at $t = 7$ (stationary solution).

STOCHASTIC NEURAL FIELD EQUATION

Sources of noise in neural fields:

- Irregularity of neuron spikes
- Non-homogeneous or irregular connectivity
- Perturbations of external stimulus

Questions we would like to answer by means of stochastic models:

- Does noise interfere in the existence of stationary solutions?
- How far the stationary solutions can be modified as a result of noise?
- May a stationary solution be transformed into another one just as a result of noise? What is the probability that this happens?

STOCHASTIC MODELS

NFE with Additive Noise:

$$dU_t(x) = \left(I(x,t) - \alpha U_t(x) + \int_\Omega K(|x-y|)S(U_t(y))dy \right) dt + \epsilon dW_t(x), \tag{2}$$

where $t \in [0,T]$, $x \in \Omega \subset \mathbb{R}^n$, W_t is a Q-Wiener process.
We consider also the following delay equation:

$$dU_t(x) = \left(I(x,t) - \alpha U_t(x) + \int_\Omega K(|x-y|)S(U_{t-\tau}(y))dy \right) dt + \epsilon dW_t(x) \tag{3}$$

τ - delay, depending on the distance $|x-y|$.

Initial condition:

$$U_t(x) = U_0(x,t), \qquad t \in [-\tau_{max}, 0], \quad x \in \Omega, \tag{4}$$

where $U_0(x,t)$ - given stochastic process, $\tau_{max} = |\Omega|/v$, v - propagation speed of the signals. We will consider domains of the form $\Omega = [-l, l]$, including the limit case when $l \to \infty$.

ONGOING WORK

RESEARCH PROJECT NEUROFIELD

Main Institution: IST-ID, University of Lisbon
Partner: University of Minho
Reference: POCI-01-0145-FEDER-031393
Duration: 3 years (started in September 2018)
Budget: 248000 E (partially supported by EU funds)
Title : Analysis and numerical simulation of deterministic and stochastic neural field equations with applications to robotics.
The project team includes 5 researchers (3 - IST, 2 -Uminho).
Two of these researchers are postdocs hired specially to work on this project.

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